Decentralized Risk-Sharing[1]

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[1] Workshop in memory of Ermanno Pitacco.

Main references[2]

- **Convex order and comonotonic conditional mean risk sharing**. Denuit M., Dhaene J. (2012). *Insurance: Mathematics and Economics,* 51, 249-256.
- **Risk-sharing rules and their properties, with applications to peer-to-peer insurance**. Denuit M., Dhaene J., Robert C.Y. (2022). *Journal of Risk and Insurance,* 89(3), 615-667.
- **Comonotonicity and Pareto optimality, with application to collaborative insurance**. Denuit M., Dhaene J., Ghossoub M., Robert C.Y. (2023). *LIDAM Discussion Paper ISBA 2023/05.*
- **An axiomatic theory for quantile-based risk sharing**. Dhaene J., Robert C.Y, Cheung K.C., Denuit M. (2023). *Submitted.*

[2] Hereafter, we will refer to these papers as D,D (2012), D,D,R (2022), D,D,G,R (2023) and D,R,C,D (2023).

Agenda

- 1. Introduction.
- 2. Risk-sharing and risk-sharing rules.
- 3. Examples of risk-sharing rules.
- 4. Internal risk-sharing rules.
- 5. Aggregate risk-sharing rules.
- 6. Properties of risk-sharing rules.
- 7. Axiomatic characterization of the CM risk-sharing rule.
- 8. Further topics on DRS, not covered in this talk.

1. Introduction

Risk-sharing pools

- Consider a pool of individual random future losses.
- **Centralized risk-sharing**:
  - Refers to **risk transfer** mechanisms where the individual losses faced by the members of the pool are passed to a central insurer.
  - Each **insured** in the **insurance portfolio** is compensated *ex-post* from the insurer for his loss.
  - In return, the insurer charges an *ex-ante* **premium** to each insured and sets up a **solvency capital** at initiation.
  - The premiums follow from an appropriate **premium principle**.
  - Premiums and solvency capital are set such that the probability that their sum at time 1 exceeds the aggregate loss of the insurance portfolio is sufficiently high.

1. Introduction

Risk-sharing pools

- Consider again a pool of individual random future losses.
- **Decentralized risk-sharing** (DRS):
  - Refers to **risk-sharing** (RS) mechanisms under which the participants in the pool share their risks among each other.
  - Each **participant** in the **risk-sharing pool** is compensated *ex-post* from the pool for his loss.
  - In return, each participant pays an *ex-post* **contribution** to the pool.
  - These contributions follow from an appropriate **risk-sharing rule**, which is chosen such that the sum of all individual contributions to the pool is equal to the aggregate loss of the pool.
  - A decentralized approach does not require setting up a solvency capital, due to the full allocation condition.

- **Risk-sharing pools of i.i.d. losses**:
  - A natural choice for i.i.d.[3] losses is the **uniform risk-sharing** rule, where each participant contributes ex-post an equal part of the aggregate loss.
- **Risk-sharing pools of losses which are not i.i.d.**:
  - In this case, the choice of an appropriate and simple RS rule is often not straightforward.
  - A possible choice is the **conditional mean RS** (CMRS) rule[4], where each participant contributes ex-post the conditional expectation of his loss brought to the pool, given the aggregate loss covered by the pool.

[3] i.i.d. = independent and identically distributed.
[4] See D,D (2012).

1. Introduction

Peer-to-peer (P2P) insurance

- **P2P insurance** (also called **collaborative insurance**, **crowdsurance** or **decentralized insurance**) refers to the risk management strategy where:
  - a group of participants bring their losses together in a RS pool,
  - the pool covers the losses of its participants,
  - against a contribution to be determined and paid ex post by each member.
- The **group of participants** who form the pool can consist of:
  - friends, family members, affinity groups, patients suffering a same disease, etc.
  - lawyers, farmers, physicians, etc., who form a risk pool to protect themselves against professional risks.
- Natural catastrophes and major industrial risks (induced by nuclear plants e.g.) are often covered by RS pools.

1. Introduction

Peer-to-peer (P2P) insurance

- P2P insurance revives **early forms of mutual insurance** (where the *contributions of the many* are used to cover the *losses of the few*).
- Risk-sharing mechanisms have been studied for decades in the actuarial literature, starting with the pioneering work of **Karl Borch** on equilibrium in reinsurance markets in the 1960's.

1. Introduction

Peer-to-peer (P2P) insurance

- The recent interest in a **sharing economy**, **collaborative consumption** and **decentralized finance/insurance**, together with recent **advances in technology** have made P2P insurance a viable candidate to partially disrupt the traditional insurance sector.
- In response to these recent evolutions, several actuarial researchers have shown a renewed interest in the math supporting P2P insurance[5].

[5] See e.g. Abdikerimova & Feng (2019), Denuit (2019), Denuit, Dhaene & Robert (2022), Feng, Liu & Zhang (2022), Jiao, Kou, Liu & Wang (2022), Feng (2023).

- Time 0 is 'now'.
- All r.v.'s are real-valued and defined on the atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- A random vector is denoted by a bold upper-case letter with subscript indicating its dimension, e.g.

$$\mathbf{X}_n = (X_1, X_2, \ldots, X_n),$$

- while its realization (observed at time 1) is denoted by the corresponding bold lower-case small letter, e.g.

$$\mathbf{x}_n = (x_1, x_2, \ldots, x_n)$$

2. Risk-sharing and risk-sharing rules

Agents and their losses

- Let χ be an appropriate (sufficiently rich) set of r.v.'s on $(\Omega, \mathcal{F}, \mathbb{P})$, representing possible random losses at time 1.[6]
- Consider n economic **agents**, numbered $i = 1, 2, \ldots, n$.
- Each agent i faces a l **loss** $X_i \in \chi$ at the end of the observation period $[0, 1]$.
- Without insurance or pooling, each agent bears his own loss:
  - At time 1, agent i suffers loss x_i, which is the realization of X_i.

[6] If not explicitely stated differently, we assume that $\chi = L^1$ or L^1_+, appropriate for the situation at hand.

- The joint cdf of the the loss vector $\mathbf{X}_n$ is denoted by $F_{\mathbf{X}_n}$.
- The marginal cdf's of the individual losses are denoted by $F_{X_1}, F_{X_2}, \ldots, F_{X_n}$, respectively.
- The **aggregate loss** faced by the n agents with loss vector $\mathbf{X}_n$ is denoted by $S_{\mathbf{X}_n} = \sum_{i=1}^{n} X_i$.
- Hereafter, we will often call $\mathbf{X}_n$ the **pool**, and call each agent a **participant** in the pool.
- In case no confusion about n is possible, we will write $\mathbf{X}$ instead of $\mathbf{X}_n$ and $S_{\mathbf{X}}$ (or S) instead of $S_{\mathbf{X}_n}$.

2. Risk-sharing and risk-sharing rules

Allocations

- **Definition**: For any pool $\mathbf{X} \in \chi^n$ with aggregate loss $S_{\mathbf{X}}$, the set $\mathcal{A}_{\mathbf{X}}$ is defined by:

$$\mathcal{A}_{\mathbf{X}} = \left\{ (Y_1, Y_2, \ldots, Y_n) \in \chi^n \mid \sum_i^n Y_i = S_{\mathbf{X}} \right\}$$

- The elements of $\mathcal{A}_{\mathbf{X}}$ are called the n-dimensional **allocations** of $\mathbf{X}$ in χ^n.

2. Risk-sharing and risk-sharing rules

Risk-sharing

- **Definition**: **Risk-sharing** in a pool $\mathbf{X} \in \chi^n$ is a two-stage process.
  - **Ex-ante step** (at time 0):
    The losses X_i in the pool are re-allocated by transforming $\mathbf{X}$ into another random vector $\mathbf{C}(\mathbf{X}) \in \mathcal{A}_{\mathbf{X}}$:
    $$\mathbf{C}(\mathbf{X}) = \left(C_1(\mathbf{X}),\ C_2(\mathbf{X}),\ \ldots,\ C_n(\mathbf{X})\right)$$
  - **Ex-post step** (at time 1):
    - Each participant i *receives from the pool* the realization of his loss X_i.
    - In return, he *pays to the pool* the realization of his re-allocated loss $C_i(\mathbf{X})$.
- **Remark**:
  - As $\mathbf{C}(\mathbf{X}) \in \mathcal{A}_{\mathbf{X}}$, risk sharing is **self-financing**:
    $$\sum_{i=1}^{n} C_i(\mathbf{X}) = \sum_{i=1}^{n} X_i$$
  - This condition is called the **full allocation condition**.

- **Remarks**:
  - For any participant i in the pool $\mathbf{X} = (X_1, X_2, \ldots, X_n)$,
    - X_i is called his **loss**, (paid by the pool).
    - $C_i(\mathbf{X})$ is called his **contribution**, (paid to the pool).
  - **Contribution vector**:

$$\mathbf{C}(\mathbf{X}) = \left(C_1(\mathbf{X}),\ C_2(\mathbf{X}),\ \ldots,\ C_n(\mathbf{X})\right)$$

- **Definition**: A **risk-sharing rule** is a mapping $\mathbf{C} : \chi^n \rightarrow \chi^n$ which transforms any pool $\mathbf{X} \in \chi^n$ into a contribution vector $\mathbf{C}(\mathbf{X}) \in \mathcal{A}_{\mathbf{X}}$:

$$\mathbf{X} \in \chi^n \rightarrow \mathbf{C}(\mathbf{X}) \in \mathcal{A}_{\mathbf{X}}$$

Example 2.1:

- Suppose that $\chi = L^1_+$.
- Consider the RS rule $\mathbf{C}^{\text{prop}} : \chi^2 \to \chi^2$ where for any $\mathbf{X} \in \chi^2$, the contribution vector $\mathbf{C}^{\text{prop}}(\mathbf{X}) \in \mathcal{A}_{\mathbf{X}} \subseteq \chi^2$ is given by

$$C_i^{\text{prop}}(\mathbf{X}) = \frac{\mathbb{E}[X_i]}{\mathbb{E}[S_{\mathbf{X}}]} S_{\mathbf{X}}, \qquad i = 1, 2$$

- **Interpretation**: Each participant contributes a fixed proportion of the total loss $S_{\mathbf{X}}$, which is in accordance with his contribution to the expected aggregate loss.
- This RS rule is called the **proportional risk-sharing rule**.

Example 2.2:

- At time 1, we flip a coin. The r.v. Z equals 0 in case of heads and 1 in case of tails.
- Consider the RS rule $\mathbf{C} : \chi^2 \rightarrow \chi^2$, which is defined such that for any pool $\mathbf{X} = (X_1, X_2)$,

$$C_1(\mathbf{X}) = \begin{cases} X_1 + X_2 & : Z = 0 \\ 0 & : Z = 1 \end{cases}$$

and

$$C_2(\mathbf{X}) = \begin{cases} 0 & : Z = 0 \\ X_1 + X_2 & : Z = 1 \end{cases}$$

- **Observations**: In order to determine the realization of $\mathbf{C}(\mathbf{X})$:
  - Knowledge of the realization of $\mathbf{X}$ is not sufficient[7].
  - Knowledge of the realizations of $X_1 + X_2$ and Z is required.

[7] In other words, $\mathbf{C}(\mathbf{X})$ is not $\mathbf{X}$-measurable, see further.

- Consider a RS rule $\mathbf{C}:\chi^n \to \chi^n$ and a pool $\mathbf{X} \in \chi^n$.
- In order to be able to determine the contribution $C_i\left(\mathbf{X}\right)$ of each participant i at time 1, one may need **different types of information**:
  - **Deterministic information** (available at time 0):
    - certain parameters (e.g. expectations of the X_i),
    - certain cdf's (e.g. the cdf's F_i of the X_i, or the cdf $F_{\mathbf{X}}$ of $\mathbf{X}$).
  - **Realizations of random quantities** (available at time 1):
    - outcomes of certain r.v.'s and random vectors (e.g. outcome of $S_{\mathbf{X}}$, or outcome of $\mathbf{X}$, or outcome of other r.v.'s)

- **At time 0**, the contribution vector $\mathbf{C}(\mathbf{X})$ is a random vector, as it depends on $\mathbf{X}$, and eventually also on other sources of randomness.
  - Knowing the realization of $\mathbf{X}$ is often not sufficient to know the realization of $\mathbf{C}(\mathbf{X})$.
  - This means that in general we don't assume that $\mathbf{C}(\mathbf{X})$ is $\mathbf{X}$-measurable.
  - In other words, in general we don't assume that there exists a function $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathbf{C}(\mathbf{X}) = \mathbf{h}(\mathbf{X})$.
- **At time 1**, the contribution vector is a deterministic vector, as the realization of any random source in $\mathbf{C}(\mathbf{X})$ is assumed to be observable at time 1.

2. Risk-sharing and risk-sharing rules

Combining risk retention, risk-sharing and risk transfer

- In order to **reduce his contribution** to the pool, a participant with initial loss Y_i may:
  - **retain** part of Y_i
  - **transfer** part of Y_i to an insurer,
  - **share** the remaining part of Y_i with the members of a pool.
- **In this presentation**:
  - X_i denotes the loss that i shares in the pool, after eventual retention and transfer of risk.
  - $C_i(\mathbf{X})$ denotes the contribution of i to the pool.
- **Example**:
  - Suppose that participant i *retains* the lower layer $[0, l_i)$ of initial loss $Y_i \geq 0$,
  - and *transfers* the upper layer $[u_i, \infty)$ of Y_i, with $u_i > l_i$ to an insurer.
  - The loss *shared* in the pool is then

$$X_i = (Y_i - l_i)_+ - (Y_i - u_i)_+$$

3. Examples of risk-sharing rules

The stand-alone risk-sharing rule

- **Definition**: $\mathbf{C} : \chi^n \to \chi^n$ is the **stand-alone RS rule** if

$$\mathbf{C}(\mathbf{X}) = \mathbf{X}, \qquad \text{for any } \mathbf{X} \in \chi^n$$

- **Interpretation**: This RS rule corresponds to the case where individuals decide not to pool their risks.

- The pool just acts as a register to collect data about individual losses, without re-allocating them among participants.

3. Examples of risk-sharing rules

The uniform risk-sharing rule

- **Definition**: The **uniform RS rule** $\mathbf{C}^{\text{uni}}$ is defined by

$$C_i^{\text{uni}}(\mathbf{X}) = \frac{S_{\mathbf{X}}}{n}, \qquad i = 1, 2, \ldots, n,$$

for any $\mathbf{X} \in \chi^n$.

- **Interpretation**: This RS rule equally distributes the aggregate loss $S_{\mathbf{X}}$ over all participants.

- It is the most simple, non-trivial and well-known RS rule.

3. Examples of risk-sharing rules

The order statistics risk-sharing rule

- The i-th **order statistic** $X_{(i)}$ of a random vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is the i-th smallest value in $\mathbf{X}$. Hence, $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$.
- **Definition**: The **order statistics RS rule** $\mathbf{C}^{\mathrm{ord}}$ is defined by

$$\mathbf{C}^{\mathrm{ord}}(\mathbf{X}) = \left(X_{(1)}, X_{(2)}, \ldots, X_{(n)}\right)$$

  for any $\mathbf{X} \in \chi^n$.
- **Interpretation**: Participants are ordered in ascending risk-bearing capacity (e.g. age), and a higher risk-bearing capacity leads to a higher contribution.

3. Examples of risk-sharing rules

The conditional mean risk-sharing rule

- **Definition**: The **conditional mean RS rule** $\mathbf{C}^{\mathrm{cm}}$ is defined by
$$C_i^{\mathrm{cm}}(\mathbf{X}) = \mathbb{E}\left[X_i \mid S_{\mathbf{X}}\right], \qquad i = 1, 2, \ldots, n,$$
for any $\mathbf{X} \in \chi^n \subseteq \left(L^1\right)^n$.

- **Interpretation**: Each participant contributes the expected value of the loss he brings to the pool, given the aggregate loss experienced by the pool.

- The CMRS rule was introduced in the actuarial literature in D,D (2012)[8]. It has many nice properties, see D,D,R (2022). Its axiomatic characterization is given in Jiao, Kou, Liu & Wang (2022).

[8] An early reference is Landsberger & Meilijson (1994).

3. Examples of risk-sharing rules

The conditional mean risk-sharing rule

- A **reshuffle** of the pool $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is a random vector $\mathbf{X}^\pi$ defined by

$$\mathbf{X}^\pi = \left(X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(n)}\right),$$

  where $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$ is a permutation of $\{1, \ldots, n\}$.

- **Interpretation**: $\mathbf{X}$ and $\mathbf{X}^\pi$ are composed of the same individual losses, but only their places are interchanged: After reshuffling, $X_{\pi(i)}$ is the new loss attributed to participant i.

- **Definition**:[9] The pool $\mathbf{X}$ is **exchangeable** in case for any reshuffle π of $\mathbf{X}$, one has that $\mathbf{X}^\pi \overset{d}{=} \mathbf{X}$.

[9] $\mathbf{X}^\pi \overset{d}{=} \mathbf{X}$ is a notation for '$\mathbf{X}^\pi$ and $\mathbf{X}$ have the same cdf'.

3. Examples of risk-sharing rules

The conditional mean risk-sharing rule

- The losses of an exchangeable pool are i.d.[10], but not necessary mutually independent.
- **Property**: In case the pool $\mathbf{X} \in \chi^n$ is exchangeable, then

$$C_i^{\mathrm{cm}}(\mathbf{X}) = \mathbb{E}\left[X_i \mid S_{\mathbf{X}}\right] = \frac{S_{\mathbf{X}}}{n}$$

  holds for any participant i.
- **Interpretation**: For any participant in an exchangeable pool, the contributions according to the conditional mean RS rule and the uniform RS rule are equal.

[10] The notation 'i.d.' stands for 'identically distributed'.

- **Definition**:

  $\mathbf{C} : \chi^n \to \chi^n$ is an **internal RS rule** if for any pool $\mathbf{X} \in \chi^n$, one has that $\mathbf{C}(\mathbf{X})$ is $\mathbf{X}$-measurable.

- **Interpretation**:

  $\mathbf{C}$ is *internal* in the sense that the randomness of any contribution vector $\mathbf{C}(\mathbf{X})$ is solely due to the randomness of the loss vector $\mathbf{X}$.

- **Example 4.1**:

$$\mathbf{C}(\mathbf{X}) = \begin{cases} \left(\frac{X_1+X_2}{2}, \frac{X_1+X_2}{2}\right) & : F_{X_1} = F_{X_2} \\ (X_1, X_2) & : F_{X_1} \neq F_{X_2} \end{cases}$$

- **Characterization**:

  $\mathbf{C}:\chi^n \to \chi^n$ is an **internal RS rule** if and only if any of the following equivalent conditions holds:

  (1) For any $\mathbf{X} \in \chi^n$, there exists a function $\mathbf{h}^{\text{int}} : \mathbb{R}^n \to \mathbb{R}^n$ such that
  $$\mathbf{C}(\mathbf{X}) = \mathbf{h}^{\text{int}}(\mathbf{X})$$

  (2) For any $\mathbf{X} \in \chi^n$, one has that
  $$\mathbf{C}(\mathbf{X}) = \mathbb{E}\left[\mathbf{C}(\mathbf{X}) \mid \mathbf{X}\right]$$

- **Important remark**:

  The function $\mathbf{h}^{\text{int}}$ may be pool-specific (see e.g. Example 4.1).

Example 2.2 (revisited):

- At time 1, we flip a coin. The r.v. Z equals 0 in case of heads and 1 in case of tails.
- Consider the RS rule $\mathbf{C} : \chi^2 \to \chi^2$, which is defined such that for any pool $\mathbf{X} = (X_1, X_2)$,

$$C_1(\mathbf{X}) = \begin{cases} X_1 + X_2 & : Z = 0 \\ 0 & : Z = 1 \end{cases}$$

and

$$C_2(\mathbf{X}) = \begin{cases} 0 & : Z = 0 \\ X_1 + X_2 & : Z = 1 \end{cases}$$

- Observations:
  - It is impossible to find for any pool $\mathbf{X}$ a function $\mathbf{h}^{\text{int}} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\mathbf{C}(\mathbf{X}) = \mathbf{h}^{\text{int}}(\mathbf{X})$.
  - This means that $\mathbf{C}$ is not an internal RS rule.

4. Internal risk-sharing rules

Examples of internal RS rules

- **Stand-alone RS**: internal,

$$\mathbf{C}(\mathbf{X}) = \mathbf{X}$$

- **Uniform RS**: internal,

$$C_i^{\text{uni}}(\mathbf{X}) = \frac{S_{\mathbf{X}}}{n}$$

- **Proportional RS**: internal,

$$C_i^{\text{prop}}(\mathbf{X}) = \frac{\mathbb{E}[X_i]}{\mathbb{E}[S_{\mathbf{X}}]} S_{\mathbf{X}}$$

- **Order statistics RS**: internal,

$$C_i^{\text{ord}}(\mathbf{X}) = X_{(i)}$$

- **Conditional mean RS**: internal,

$$C_i^{\text{cm}}(\mathbf{X}) = \mathbb{E}\left[X_i \mid S_{\mathbf{X}}\right]$$

Example 4.2:

- Consider the RS rule $\mathbf{C} : \chi^n \to \chi^n$, where any $\mathbf{X} \in \chi^n$ is a pool of **health-related costs** of the participants.
- Suppose the participants can be divided in m **age categories**, denoted by $1, 2, \ldots, m$:
  - For any pool $\mathbf{X} \in \chi^n$, the age category of participant i is known at time 0, and denoted by $a_i(\mathbf{X})$.
- The **RS rule** $\mathbf{C}$ is defined by

$$C_i(\mathbf{X}) = \sum_{j=1}^{m} \left(\frac{\sum_{k=1}^{n} X_k \times 1_{a_k(\mathbf{X})=j}}{\sum_{k=1}^{n} 1_{a_k(\mathbf{X})=j}} \right) \times 1_{a_i(\mathbf{X})=j}$$

Example 4.2 (cont'd):

- The **RS rule** $\mathbf{C}$ is defined by

$$C_i\left(\mathbf{X}\right) = \sum_{j=1}^{m} \left(\frac{\sum_{k=1}^{n} X_k \times 1_{a_k(\mathbf{X})=j}}{\sum_{k=1}^{n} 1_{a_k(\mathbf{X})=j}} \right) \times 1_{a_i(\mathbf{X})=j}$$

- **Interpretation**: Losses are uniformly shared within each age category.

- **Observations**:
  - $\mathbf{C}$ is an **internal RS rule**.
  - $\mathbf{C}$ is in general **not** a **type I - internal RS rule**.

- **Definition**:

  RS rule $\mathbf{C} : \chi^n \to \chi^n$ is an **aggregate RS rule** if for any pool $\mathbf{X} \in \chi^n$, one has that $\mathbf{C}(\mathbf{X})$ is $S_{\mathbf{X}}$-measurable.

- **Interpretation**:

  $\mathbf{C}$ is *aggregate* in the sense that the randomness of any contribution vector $\mathbf{C}(\mathbf{X})$ is solely due to the randomness of aggregate claims $S_{\mathbf{X}}$.

- **Remark**:

  An aggregate RS rule is **anonymous,** in the sense that it doesn't care about the 'origin' of the aggregate loss $S_{\mathbf{X}}$.

- **Example 5.1**:

$$\mathbf{C}(\mathbf{X}) = \begin{cases} \left(\frac{1}{3} S_{\mathbf{X}}, \frac{2}{3} S_{\mathbf{X}}\right) & : \mathbb{E}[X_1] \leq \mathbb{E}[X_2] \\ \left(\frac{2}{3} S_{\mathbf{X}}, \frac{1}{3} S_{\mathbf{X}}\right) & : \mathbb{E}[X_1] > \mathbb{E}[X_2] \end{cases}$$

5. Aggregate risk-sharing rules

Definition

- **Characterization**:

  $\mathbf{C} : \chi^n \to \chi^n$ is an **aggregate RS rule** if and only if any of the following equivalent conditions holds:

  (1) For any $\mathbf{X} \in \chi^n$, there exists a function $\mathbf{h}^{\text{aggr}} : \mathbb{R} \to \mathbb{R}^n$ such that
  $$\mathbf{C}(\mathbf{X}) = \mathbf{h}^{\text{aggr}}(S_{\mathbf{X}})$$

  (2) For any $\mathbf{X} \in \chi^n$, one has that
  $$\mathbf{C}(\mathbf{X}) = \mathbb{E}\left[\mathbf{C}(\mathbf{X}) \mid S_{\mathbf{X}}\right]$$

- **Important remark**:

  $\mathbf{h}^{\text{aggr}}$ may be pool-specific (see e.g. Example 5.1).

5. Aggregate risk-sharing rules

Definition

- **Property**:
  An **aggregate** RS rule is an **internal** RS rule.

- Proof:
  - Let $\mathbf{C} : \chi^n \to \chi^n$ be an an aggregate RS rule and $\mathbf{X} \in \chi^n$.
  - There exists a function $\mathbf{h}^{\text{aggr}} : \mathbb{R} \to \mathbb{R}^n$ such that

$$\mathbf{C}(\mathbf{X}) = \mathbf{h}^{\text{aggr}}(S_{\mathbf{X}})$$

  - Define the function $\mathbf{h}^{\text{int}} : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\mathbf{h}^{\text{int}}(\mathbf{x}) = \mathbf{h}^{\text{aggr}}(x_1 + \ldots + x_n)$$

  - Then we find that

$$\mathbf{C}(\mathbf{X}) = \mathbf{h}^{\text{int}}(\mathbf{X})$$

  - We can conclude that $\mathbf{C}$ is an internal RS rule. $\bigtriangledown$

5. Aggregate risk-sharing rules

Examples

- **Stand-alone RS**: not aggregate,

$$\mathbf{C}(\mathbf{X}) = \mathbf{X}$$

- **Uniform RS**: **aggregrate**,

$$C_i^{\text{uni}}(\mathbf{X}) = \frac{S_{\mathbf{X}}}{n}$$

- **Proportional RS**: **aggregate**,

$$C_i^{\text{prop}}(\mathbf{X}) = \frac{\mathbb{E}[X_i]}{\mathbb{E}[S_{\mathbf{X}}]} S_{\mathbf{X}}$$

- **Order statistics RS**: not aggregate,

$$C_i^{\text{ord}}(\mathbf{X}) = X_{(i)}$$

- **Conditional mean RS**: **aggregate**,

$$C_i^{\text{cm}}(\mathbf{X}) = \mathbb{E}\left[X_i \mid S_{\mathbf{X}}\right]$$

Example 5.2:

- Consider the RS rule $\mathbf{C}: \chi^2 \to \chi^2$ where any $\mathbf{X} \in \chi^2$ is a pool of **health-related costs** of 2 participants.
- Suppose the participants can be divided in m **age categories**, denoted by $1, 2, \ldots, m$:
  - For any $\mathbf{X} \in \chi^2$, the age category of participant i is denoted by $a_i(\mathbf{X})$.
- The **RS rule** $\mathbf{C}$ is defined by

$$\mathbf{C}(\mathbf{X}) = \begin{cases} \left(\frac{S_{\mathbf{X}}}{2}, \frac{S_{\mathbf{X}}}{2}\right) & : \text{if } a_1(\mathbf{X}) = a_2(\mathbf{X}) \\ \left(\mathbb{E}\left[X_1 \mid S_{\mathbf{X}}\right], \mathbb{E}\left[X_2 \mid S_{\mathbf{X}}\right]\right) & : \text{if } a_1(\mathbf{X}) \neq a_2(\mathbf{X}) \end{cases}$$

- **Observations**:
  - $\mathbf{C}$ is an aggregate RS rule:

$$\mathbf{C}(\mathbf{X}) = \mathbf{h}^{\text{aggr}}(S_{\mathbf{X}})$$

  - $\mathbf{h}^{\text{aggr}}$ is pool-specific as it depends on $(a_1(\mathbf{X}), a_2(\mathbf{X}))$.

- **Premium principles:** have been studied in detail around the 'eighthies'[11].
- **Risk measures:** During the the 'nineties' and beyond, risk measures and their axiomatic characterizations have been studied extensively[12].
- **Risk sharing schemes**: Recently, the study of decentralized RS has (re-)gained a great interest.
  - Hereafter, we present a non-exhaustive list of properties that RS rules should/could satisfy[13].
  - Some properties are inspired by properties of premium principles or risk measures, other are tailored to RS rules.

[11] Goovaerts, De Vijlder & Haezendonck (1984) is a key reference.
[12] Artzner, Delbaen, Eber & Heath (1997) is a fundamental paper.
[13] See D,D,R (2022).

6. Properties of risk-sharing rules

Classes of properties

- Conservation properties: also hold for the stand-alone RS rule: **reshuffling**, **normalization**, **translativity**, **positive homogeneity**, **constancy**, **no rip-off**, **actuarial fairness**, **law invariance**.
- Improvement properties: guarantee that the RS rule 'improves' the situation compared to the stand-alone situation: **willingness-to-join**, **comonotonicity**.
- Coalition properties: guarantee that participants are not affected by coalitions set up between other participants: **coalition fairness**, **merging fairness**.
- Specific pool properties: should hold (or not) for specific pools only: **stand-alone property for comonotonic pools**, **uniformity property for exchangeable pools**.

6.1. Conservation properties of risk-sharing rules

The 'reshuffling' property

- **Definition**:[14] A RS rule $\mathbf{C} : \chi^n \to \chi^n$ satisfies the **reshuffling** property if for any $\mathbf{X} \in \chi^n$ and any of its reshuffles $\mathbf{X}^{\pi}$, one has that

$$C_i\left(\mathbf{X}^{\pi}\right) = C_{\pi(i)}\left(\mathbf{X}\right) \qquad \text{for any } i = 1, \ldots, n \tag{1}$$

- **Example**:
  - Consider $\mathbf{X} = (X_1,\ X_2,\ X_3)$ and $\mathbf{X}^{\pi} = (X_3,\ X_1,\ X_2)$.
  - If (1) holds, then

$$\pi(1) = 3 \Rightarrow C_1\left(\mathbf{X}^{\pi}\right) = C_3\left(\mathbf{X}\right)$$

- **Interpretation** : If participants exchange their individual losses, then their contributions are changed in the same way.

[14] Jiao, Yiao, Liu & Wang (2022), hereafter abbreviated as J,K,L,W (2022), call this property *symmetry*.

6.1. Conservation properties of risk-sharing rules

The 'reshuffling' property

- **Order statistics RS**: **does not satisfy reshuffling**,

$$C_1^{\text{ord}}\left(\mathbf{X}^{\pi}\right) = C_1^{\text{ord}}\left(X_3, X_2, X_1\right) = \min\left(X_1, X_2, X_3\right)$$

while

$$C_{\pi(1)}^{\text{ord}}\left(\mathbf{X}\right) = C_3^{\text{ord}}\left(X_1, X_2, X_3\right) = \max\left(X_1, X_2, X_3\right)$$

- **Conditional means RS**: **satisfies reshuffling**,

$$C_i^{\text{cm}}\left(\mathbf{X}^{\pi}\right) = \mathbb{E}\left[X_{\pi(i)} \mid S\right] = C_{\pi(i)}^{\text{cm}}\left(\mathbf{X}\right)$$

- **Definition**:[15] A RS rule $\mathbf{C} : \chi^n \to \chi^n$ satisfies the **normalization** property if for any $\mathbf{X} \in \chi^n$, one has

$$X_i = 0 \Rightarrow C_i\left(\mathbf{X}\right) = 0$$

holds for any i.

- **Interpretation**: A participant with a zero loss has a zero contribution.

- **Remark**: If a RS rule satisfies the reshuffling property, then a sufficient condition for the normalization property to hold is that it holds for the last participant.

[15] J,K,L,W (2022) call this property *zero preserving*. Normalization as defined here is a weaker property than normalization defined in D,D,R (2022).

6.1. Conservation properties of risk-sharing rules

The 'normalization' property

- **Uniform RS**: **does not satisfy normalization**,

$$C_3^{\text{uni}}\left(X_1, X_2, 0\right) = \frac{X_1 + X_2}{3}$$

- **Conditional means RS**: **satisfies normalization**,
  - The CMRS satisfies reshuffling.
  - Furthermore,

$$C_n^{\text{cm}}\left(X_1, X_2, \dots, X_{n-1}, 0\right) = \mathbb{E}\left[0 \mid S\right] = 0$$

6.1. Conservation properties of risk-sharing rules

The 'translativity' property

- **Notation**: $\mathbf{e}_j$ is the n-dimensional unit vector with all components equal to 0, except the j-th component which equals 1.
- **Definition**: A RS rule $\mathbf{C} : \chi^n \to \chi^n$ satisfies the **translativity** property if for any $\mathbf{X} \in \chi^n$ and any participant $j = 1, \ldots, n$, one has that

$$C_i\left(\mathbf{X} + c\ \mathbf{e}_j\right) = C_i\left(\mathbf{X}\right), \qquad \text{for all } i \neq j \text{ and any } c \geq 0$$

- **Interpretation**: Increasing the loss of participant j by a deterministic amount $c \geq 0$, leaves the contributions of the other participants unchanged.
- **Consequence**: In case the RS rule $\mathbf{C}$ satisfies the translativity property, then

$$C_j\left(\mathbf{X} + c\ \mathbf{e}_j\right) = C_j\left(\mathbf{X}\right) + c$$

6.1. Conservation properties of risk-sharing rules

The 'translativity' property

- **Remark**: If the RS rule satisfies the reshuffling property, then a sufficient condition for the translativity property to hold is that it holds for the last participant.

- **Uniform RS**: **does not satisfy translativity**,

$$C_1^{\text{uni}}(X_1, X_2, X_3 + c) = \frac{S + c}{3} \neq \frac{S}{3} = C_1^{\text{uni}}(X_1, X_2, X_3)$$

- **Conditional mean RS** **satisfies translativity**,
  - The CMRS satisfies reshuffling.
  - For any $i \neq n$, one has

$$C_i^{\text{cm}}(\mathbf{X} + c\ \mathbf{1}_n) = \mathbb{E}[X_i \mid |S + c] = \mathbb{E}[X_i \mid S] = C_i^{\text{cm}}(\mathbf{X})$$

6.1. Conservation properties of risk-sharing rules

The 'positive homogeneity' property

- **Definition**:
  A RS rule $\mathbf{C} : \chi^n \rightarrow \chi^n$ satisfies the **positive homogeneity** property[16] if for any $\mathbf{X} \in \chi^n$, one has that

$$C_i \left(c \ \mathbf{X} \right) = c \times C_i \left(\ \mathbf{X} \right)$$

  holds for any i and any $c \geq 0$.

- **Interpretation**: Multiplying all losses by $c \geq 0$ implies that all contributions change in the same way.

[16] $c \ \mathbf{X}$ stands for $\left(c \times X_1, c \times X_2, \ldots, c \times X_n \right)$

- **Proportional RS is positive homogeneous**,

$$C_i^{\text{prop}}(c\,\mathbf{X}) = \frac{\mathbb{E}[c\,X_i]}{\mathbb{E}[c\,S]}(c \times S) = c\frac{\mathbb{E}[X_i]}{\mathbb{E}[S]}S = c \times C_i^{\text{prop}}(\mathbf{X})$$

- **Order statistics RS is positive homogeneous**,

$$C_i^{\text{ord}}(c\,\mathbf{X}) = c \times X_{(i)} = c \times C_i^{\text{ord}}(\mathbf{X})$$

- **Conditional mean RS is positive homogeneous**,

$$C_i^{\text{cm}}(c\,\mathbf{X}) = \mathbb{E}[c \times X_i \mid c \times S] = c\,\mathbb{E}[X_i \mid S] = c \times C_i^{\text{cm}}(\mathbf{X})$$

6.1. Conservation properties of risk-sharing rules

The 'constancy' property

- **Definition**:[17] A RS rule $\mathbf{C} : \chi^n \rightarrow \chi^n$ satisfies the **constancy** property if for any $\mathbf{X} \in \chi^n$, one has

$$X_j = c \Rightarrow C_i(\mathbf{X}) = c$$

holds for any j and any $c \geq 0$.

- **Interpretation**: In case a participant's loss is a non-negative constant, then his contribution is equal to that constant.

- **Remark**: If the RS rule satisfies the reshuffling property, then a sufficient condition for the constancy property to hold is that it holds for the last participant.

[17] Constancy as defined here is a weaker property than the constancy defined in D,D,R (2022). Constancy is called *constant preserving* in J,K,L,W (2022).

6.1. Conservation properties of risk-sharing rules

The 'constancy' property

- **Uniform RS**: **does not satisfy constancy**,

$$C_3^{\text{uni}}(X_1, X_2, c) = \frac{X_1 + X_2 + c}{3}$$

- **Proportional RS**: **does not satisfy constancy**,

$$C_3^{\text{prop}}(X_1, X_2, c) = \frac{c}{\mathbb{E}[S]} S$$

- **Conditional mean RS**: **satisfies constancy**,

$$C_n^{\text{cm}}(X_1, X_2, \ldots, X_{n-1}, c) = \mathbb{E}[c \mid S] = c$$

- **Proposition**:
  If a RS rule satisfies **translativity** and **normalization**, then it also satisfies **constancy**.

- **Proof**:
  - The case where $X_n = c$:

$$C_n\big(X_1, X_2, \ldots, X_{n-1}, c\big) = C_n\big(X_1, X_2, \ldots, X_{n-1}, 0 + c\big) \overset{\text{transl.}}{=} C_n\big(X_1, X_2, \ldots, X_{n-1}, 0\big) + c \overset{\text{normaliz.}}{=} c$$

  - The case where $X_j = c$ for $j \neq n$ is proven similarly. $\bigtriangledown$

- **The 'largest value' of r.v. X**:

$$F_X^{-1}(1) = \inf\{x \in \mathbb{R} \mid F_X(x) = 1\}$$

- **Definition**:[18] A RS rule $\mathbf{C} : \chi^n \to \chi^n$ satisfies the **no rip-off** property if for any $\mathbf{X} \in \chi^n$, one has that

$$C_i(\mathbf{X}) \leq F_{X_i}^{-1}(1)$$

holds for any $i = 1, 2, \ldots, n$.

- **Interpretation**: The participant's contribution will never exceed his 'worst-case loss'.

[18] J,K,L,W (2022) call this property the 'risk fairness' axiom.

Uniform RS: **does not satisfy no-ripoff**,[19]

$$C_1^{\text{uni}}(U, 2U) = \frac{3U}{2} \in [0, 1.5]$$

Proportional RS: **does not satisfy no-ripoff**,

$$C_3^{\text{prop}}(X_1, X_2, c) = \frac{c}{\mathbb{E}[S]} S$$

Conditional mean RS: **satisfies no rip-off**,

$$C_i^{\text{cm}}(\mathbf{X}) = \mathbb{E}[X_i \mid S] \leq \mathbb{E}[F_{X_i}^{-1}(1) \mid S] = F_{X_i}^{-1}(1) \, .$$

[19] In this text, the notation U is used exclusively for a r.v. which is uniformly distributed over $[0, 1]$.

6.1. Conservation properties of risk-sharing rules

The 'actuarial fairness' property

- **Definition**: A RS rule $\mathbf{C} : \chi^n \rightarrow \chi^n$ satisfies **actuarial fairness** if for any $\mathbf{X} \in \chi^n$, one has that

$$\mathbb{E}\left[C_i\left(\mathbf{X}\right)\right] = \mathbb{E}\left[X_i\right]$$

  holds for any $i = 1, 2, \ldots, n$.

- **Interpretation**: On average, participants do neither gain nor lose from risk sharing, in the sense that their expected contribution (by joining the pool) is equal to their expected loss (when staying alone).

6.1. Conservation properties of risk-sharing rules

The 'actuarial fairness' property

- **Uniform RS**: **does not satisfy actuarial fairness**,
  - Consider the loss vector $\big(U, U, 4U\big)$.
  - Expected loss of first participant:

$$\mathbb{E}[U] = \frac{1}{2}$$

  - Expected contribution of first participant:

$$\mathbb{E}\big[C_1^{\text{uni}}\big(U, U, 4U\big)\big] = \mathbb{E}\big[\frac{U + U + 4U}{3}\big] = 1$$

- **Conditional mean RS**: **satisfies actuarial fairness**,

$$\mathbb{E}\left[C_i^{\text{cm}}\left(\mathbf{X}\right)\right] = \mathbb{E}\big[\mathbb{E}[X_i \mid S]\big] = \mathbb{E}\left[X_i\right]$$

6.1. Conservation properties of risk-sharing rules

The 'actuarial fairness' property

- **Proposition**:
  If a RS rule satisfies **actuarial fairness** and **no rip-off**, then it also satisfies **constancy**.

- **Proof**
  - Suppose $X_i = c$.
  - Actuarial fairness implies

$$\mathbb{E}\left[C_i\left(\mathbf{X}\right)\right] = \mathbb{E}\left[X_i\right] = c$$

  - No rip-off implies

$$C_i\left(\mathbf{X}\right) \leq F_{X_i}^{-1}(1) = c$$

  - We can conclude that

$$C_i\left(\mathbf{X}\right) = c$$

$\bigtriangledown$

6.1. Conservation properties of risk-sharing rules

The 'law-invariance' property

- **Definition**:

  A RS rule $\mathbf{C} : \chi^n \to \chi^n$ satisfies the **law-invariance** property in case

  $$\mathbf{X} \stackrel{\mathrm{d}}{=} \mathbf{Y} \Rightarrow \mathbf{C}\left(\mathbf{X}\right) \stackrel{\mathrm{d}}{=} \mathbf{C}\left(\mathbf{Y}\right)$$

  holds for any $\mathbf{X}$ and $\mathbf{Y} \in \chi^n$.

- **Interpretation**:

  If pools are 'equal in distribution', then also their contribution vectors are 'equal in distribution'.

- **Proportional RS**: **is law invariant**: If $\mathbf{X} \overset{d}{=} \mathbf{Y}$, then

$$\left(\frac{\mathbb{E}[X_1]}{\mathbb{E}[S_{\mathbf{X}}]}S_{\mathbf{X}}, \dots, \frac{\mathbb{E}[X_n]}{\mathbb{E}[S_{\mathbf{X}}]}S_{\mathbf{X}}\right) \overset{d}{=} \left(\frac{\mathbb{E}[Y_1]}{\mathbb{E}[S_{\mathbf{Y}}]}S_{\mathbf{Y}}, \dots, \frac{\mathbb{E}[Y_n]}{\mathbb{E}[S_{\mathbf{Y}}]}S_{\mathbf{Y}}\right)$$

- **Conditional means RS**: **is law invariant**: If $\mathbf{X} \overset{d}{=} \mathbf{Y}$, then

$$\mathbb{E}\left[\mathbf{X} \mid S_{\mathbf{X}}\right] \overset{d}{=} \mathbb{E}\left[\mathbf{Y} \mid S_{\mathbf{Y}}\right]$$

Example 6.1.1:

- Suppose that U and V are i.i.d. r.v.'s in χ which are uniformly distributed over the unit interval.

- Consider the RS rule $\mathbf{C} : \chi^2 \rightarrow \chi^2$ defined by

$$\mathbf{C}(\mathbf{X}) = \begin{cases} (X_1, X_2) & : \text{ if } X_1 = U \\ \left(\frac{X_1+X_2}{2}, \frac{X_1+X_2}{2}\right) & : \text{ otherwise} \end{cases}$$

- **Observation**:

$$(U, 1-U) \overset{\text{d}}{=} (V, 1-V) \text{ but } \mathbf{C}(U, 1-U) \overset{\text{d}}{\neq} \mathbf{C}(V, 1-V)$$

- **Conclusion**: **C** is not 'law-invariant' in the sense that there exist pools **X** and **Y**, such that

$$\mathbf{X} \overset{\text{d}}{=} \mathbf{Y} \text{ and } \mathbf{C}(\mathbf{X}) \overset{\text{d}}{\neq} \mathbf{C}(\mathbf{Y})$$

- **Conservation properties**: also hold for the stand-alone RS rule: **reshuffling**, **normalization**, **translativity**, **positive homogeneity**, **constancy**, **no rip-off**, **actuarial fairness**, **law invariance**.
- **Improvement properties**: guarantee that the RS rule 'improves' the situation compared to the stand-alone situation: **willingness-to-join**, **comonotonicity**.
- **Coalition properties**: guarantee that participants are not affected by coalitions between other participants **coalition fairness**, **merging fairness**.
- **Specific pool properties**: should hold (or not) for specific pools only: **stand-alone property for comonotonic pools**, **uniformity property for exchangeable pools**.

6.2. Improvement properties of ris- sharing rules

Stop-loss order

- Consider the r.v.'s X and Y.

- **Definition**:
  X is smaller than Y in **stop-loss order** if for any non-decreasing concave function $u : \mathbb{R} \to \mathbb{R}$ and any real w, one has that

$$\mathbb{E}[u(w - X)] \geq \mathbb{E}[u(w - Y)],$$

  provided the expectations exist.

- **Interpretation in expected utility theory**:
  Risk-averse decision makers prefer loss X over loss Y.

- **Notation**:

$$X \leq_{\text{sl}} Y$$

- **Definition**:
  A RS rule $\mathbf{C} : \chi^n \to \chi^n$ satisfies the **willingness-to-join** property[20] if for any $\mathbf{X} \in \chi^n$, one has that

$$C_i(\mathbf{X}) \leq_{\text{sl}} X_i$$

  holds for any $i = 1, 2, \ldots, n$.

- **Interpretation**: Any risk-averse participant i will prefer the contribution $C_i(\mathbf{X})$ over his initial loss X_i.

[20] In J,K,L,W (2022) this property is called the *universal improvement property*.

6.2. Improvement properties of risk-sharing rules

The 'willingness-to-join' property

- **Proposition**: If a RS rule satisfies **willingness-to-join,** then it also satisfies **actuarial fairness**, **no rip-off** and **constancy**.
- **Proof**:
  - For any i, willingness-to-join implies
    $$\mathbb{E}\left[C_i\left(\mathbf{X}\right)\right] \leq \mathbb{E}\left[X_i\right]$$
    The full allocation condition states that
    $$\sum_{i=1}^{n} \mathbb{E}\left[C_i\left(\mathbf{X}\right)\right] = \sum_{i=1}^{n} \mathbb{E}\left[X_i\right]$$
    We can conclude that $\mathbf{C}$ is actuarially fair:
    $$\mathbb{E}\left[C_i\left(\mathbf{X}\right)\right] = \mathbb{E}\left[X_i\right]$$
  - Willingness-to-join implies no rip-off:
    $$C_i\left(\mathbf{X}\right) \leq F^{-1}_{C_i(\mathbf{X})}\left(1\right) \leq F^{-1}_{X_i}\left(1\right)$$
  - Actuarial fairness and no rip-off imply constancy. $\triangledown$

6.2. Improvement properties of risk-sharing rules

The 'willingness-to-join' property

- **Proportional RS**: **does not satisfy willingness-to-join**, since it does not satisfy the no rip-off property.

- **Order statistics RS**: **does not satisfy willingness-to-join**, since it is not an actuarially fair RS rule.

- **Conditional mean RS**: **satisfies willingness-to-join**:

$$\mathbb{E}[X_i \mid S] \leq_{\text{cx}} X_i$$

- Consider the pool $\mathbf{X} = (X_1, \ldots, X_n)$ with aggregate loss $S_{\mathbf{X}}$.
- **Definition**: $\mathbf{X}$ is **comonotonic** if there exist non-decreasing functions $g_i : \mathbb{R} \to \mathbb{R}$ such that

$$\mathbf{X} = \big(g_1(S_{\mathbf{X}}), \ldots, g_n(S_{\mathbf{X}})\big)$$

- **Interpretation**: $\mathbf{X}$ is comonotonic if the increase of one of the individual losses implies an increase of all the other individual losses.

- **Definition**: A RS rule $\mathbf{C} : \chi^n \rightarrow \chi^n$ is **comonotonic** if for any $\mathbf{X} \in \chi^n$, one has that $\mathbf{C}(\mathbf{X})$ is a comonotonic random vector.

- **Equivalent definition**: A RS rule $\mathbf{C}$ is **comonotonic** if for any $\mathbf{X} \in \chi^n$ there exists a function $\mathbf{h}^{\text{com}} : \mathbb{R} \rightarrow \mathbb{R}^n$, such that

$$\mathbf{C}(\mathbf{X}) = \mathbf{h}^{\text{com}}(S_{\mathbf{X}}) = \left(h_1^{\text{com}}(S_{\mathbf{X}}), \ldots, h_n^{\text{com}}(S_{\mathbf{X}})\right)$$

  where all functions h_i^{com} are non-decreasing.

- **Interpretation**: Comonotonicity of a RS rule ensures that any participant has an interest in keeping his loss as small as possible[21].

[21] For this reason, Carlier & Dana (2003) call this property the *no-sabotage condition*.

- **Remark 1**:

$$\mathbf{C} \text{ is } \textbf{comonotonic} \Rightarrow \mathbf{C} \text{ is } \textbf{aggregate} \Rightarrow \mathbf{C} \text{ is } \textbf{internal}$$

- **Remark 2**:

$$\mathbf{C} \text{ is internal} \not\Rightarrow \mathbf{C} \text{ is aggregate} \not\Rightarrow \mathbf{C} \text{ is comonotonic}$$

- **Stand-alone RS**: is **not comonotonic**. Indeed, for any $\mathbf{X}$, one has that

$$\mathbf{C}(\mathbf{X}) = \mathbf{X}$$

which is in general not comonotonic.

- **Uniform RS**: **comonotonic**. Indeed, for any $\mathbf{X}$, one has that $\mathbf{C}^{\text{uni}}(\mathbf{X})$ is comonotonic:

$$\mathbf{C}^{\text{uni}}(\mathbf{X}) = \left(\frac{S_{\mathbf{X}}}{n}, \ldots, \frac{S_{\mathbf{X}}}{n}\right)$$

- **Conditional mean RS**: **not comonotonic**, since the components of the contribution vector

$$\mathbf{C}^{\text{cm}}(\mathbf{X}) = \left(\mathbb{E}[X_1 \mid S_{\mathbf{X}}], \ldots, \mathbb{E}[X_n \mid S_{\mathbf{X}}]\right)$$

are not necessarily non-decreasing in $S_{\mathbf{X}}$.

Example 6.2.1:

- Consider the RS rule $\mathbf{C} : \chi^n \rightarrow \chi^n$.
- Suppose that each participant of any pool $\mathbf{X}$ belongs to category 0 ('poor') or 1 ('rich').
- For any pool $\mathbf{X}$, the wealth-status (0 or 1) of any participant i is known at time 0, and denoted by $w_i(\mathbf{X})$.
- The **RS rule** $\mathbf{C}$ is defined by

$$C_i(\mathbf{X}) = \frac{S_{\mathbf{X}}}{\sum_{k=1}^{n} w_k(\mathbf{X})} \times w_i(\mathbf{X})$$

in case at least one participant is 'rich', and by

$$C_i(\mathbf{X}) = \frac{S_{\mathbf{X}}}{n}$$

if all participant are 'poor'.

6.2. Improvement properties of risk-sharing rules

Comononotonic does not imply type - I comonotonic

Example 6.2.1 (cont'd):

- The **RS rule** **C** is defined by

$$C_i\left(\mathbf{X}\right) = \frac{S_{\mathbf{X}}}{\sum_{k=1}^{n} w_k\left(\mathbf{X}\right)} \times w_i\left(\mathbf{X}\right)$$

  in case at least one participant is 'rich', and by

$$C_i\left(\mathbf{X}\right) = \frac{S_{\mathbf{X}}}{n}$$

  if all participants are 'poor'.

- **Interpretation**: The aggregate losses of all participants are uniformly shared between the 'rich'.

- **Observations**:
  - **C** is a comonotonic RS rule.
  - **C** is not a type I - comonotonic RS rule (in case $F_{\mathbf{X}}$ does not uniquely determine the wealth level of all participants).

6.3. Coalition properties of risk-sharing rules

Classes of properties

- **Conservation properties**: also hold for the stand-alone RS rule: **reshuffling**, **normalization**, **translativity**, **positive homogeneity**, **constancy**, **no rip-off**, **actuarial fairness**, **law invariance**.

- **Improvement properties**: guarantee that the RS rule 'improves' the situation compared to the stand-alone situation: **willingness-to-join**, **comonotonicity**.

- **Coalition properties**: guarantee that participants are not affected by coalitions between other participants: **coalition fairness**, **merging fairness**.

- **Specific pool properties**: should hold (or not) for specific pools only: **stand-alone property for comonotonic pools**, **uniformity property for exchangeable pools**.

6.3. Coalition properties of risk-sharing rules

The 'coalition fairness' property

- Suppose that the participants in the pool $\left(X_1, X_2, X_3\right)$ agree to use the RS rule **C** to determine their contributions.
- Next, suppose that participants 2 and 3 decide to form a **coalition**, under which they equally share their joint loss $\left(X_2 + X_3\right)$.
- This coalition transforms the pool $\left(X_1, X_2, X_3\right)$ into

$$\left(X_1, \frac{X_2 + X_3}{2}, \frac{X_2 + X_3}{2}\right)$$

- It seems reasonable to require that the contribution of participant 1, who is excluded from this coalition, remains unchanged.
- RS rules satisfying this property will be said to satisfy the **coalition fairness** property.

- **Recall**: An **allocation** $\mathbf{Y}$ of a pool $\mathbf{X} \in \chi^n$ is an element of the following set:

$$\mathcal{A}_{\mathbf{X}} = \left\{ (Y_1, Y_2, \ldots, Y_n) \in \chi^n \mid \sum_i^n Y_i = S_{\mathbf{X}} \right\}$$

- A **coalition** between 2 or more partipants in the pool $\mathbf{X}$, excluding participant i, transforms $\mathbf{X}$ into an allocation $\mathbf{Y} \in \mathcal{A}_{\mathbf{X}}$, with $Y_i = X_i$.

- **Definition**:
  A RS rule $\mathbf{C} : \chi^n \to \chi^n$ satisfies the **coalition fairness** property if for any $\mathbf{X} \in \chi^n$ and any $\mathbf{Y} \in \mathcal{A}_{\mathbf{X}}$, one has that

$$Y_i = X_i \Rightarrow C_i(\mathbf{Y}) = C_i(\mathbf{X})$$

  holds for any i.

- **Interpretation**:
  - The contribution of a participant in a pool is not affected by coalitions between other participants in this pool.
  - A coalition between a group of participants does not change the joint contributions of this group.

- Consider the pools $\mathbf{X}$ and $\mathbf{Y}$ in $\mathcal{A}_{\mathbf{X}}$, with $Y_i = X_i$.
- **Uniform RS**: is **coalition fair**. Indeed,

$$C_i^{\text{uni}}(\mathbf{Y}) = \frac{S_{\mathbf{Y}}}{n} = \frac{S_{\mathbf{X}}}{n} = C_i^{\text{uni}}(\mathbf{X})$$

- **Proportional RS**: is **coalition fair**. Indeed,

$$C_i^{\text{prop}}(\mathbf{Y}) = \frac{\mathbb{E}[Y_i]}{\mathbb{E}[S_{\mathbf{Y}}]} S_{\mathbf{Y}} = \frac{\mathbb{E}[X_i]}{\mathbb{E}[S_{\mathbf{X}}]} S_{\mathbf{X}} = C_i^{\text{prop}}(\mathbf{X})$$

6.3. Coalition properties of risk-sharing rules

The 'coalition fairness' property

- **Order statistics RS**: is **not coalition fair**. Indeed,
  - Consider $\left(X_1, X_2, X_3\right)$ and $\left(X_1, X_2 + X_3, 0\right)$.
  - One has that

$$C_1^{\text{ord}}\left(X_1, X_2, X_3\right) = \min\left\{X_1, X_2, X_3\right\}$$

  - while

$$C_1^{\text{ord}}\left(X_1, X_2 + X_3, 0\right) = \min\left\{X_1, X_2 + X_3, 0\right\}$$

- **Conditional Mean RS**: is **coalition fair**. Indeed,

$$C_i^{\text{cm}}\left(\mathbf{Y}\right) = \mathbb{E}\left[Y_i \mid S_{\mathbf{Y}}\right] = \mathbb{E}\left[X_i \mid S_{\mathbf{X}}\right] = C_i^{\text{cm}}\left(\mathbf{X}\right)$$

6.3. Coalition properties of risk-sharing rules

The 'coalition fairness' property

- **Proposition**:

  A RS rule $\mathbf{C} : \chi^n \rightarrow \chi^n$ satisfies **coalition fairness**

  $\Leftrightarrow$ for any $\mathbf{X} \in \chi^n$ and any participant i, one has that

  $$C_i\left(\mathbf{X}\right) = C_i\big(X_i \ \mathbf{e}_i + \left(S_{\mathbf{X}} - X_i\right) \ \mathbf{e}_j\big) \qquad \text{for any } j \neq i \quad (2)$$

- **Proposition**:
  If a RS rule satisfies **normalization** and **coalition fairness**, then it also satisfies **reshuffling**.

6.4. Specific pool properties of risk-sharing rules

- **Conservation properties**: also hold for the stand-alone RS rule: **reshuffling**, **normalization**, **translativity**, **positive homogeneity**, **constancy**, **no rip-off**, **actuarial fairness**, **law invariance**.

- **Improvement properties**: guarantee that the RS rule 'improves' the situation compared to the stand-alone situation: **willingness-to-join**, **comonotonicity**.

- **Coalition properties**: guarantee that participants are not affected by coalitions between other participants: **coalition fairness**, **merging fairness**.

- **Specific pool properties**: should hold (or not) for specific pools only: **stand-alone property for comonotonic pools**, **uniformity property for exchangeable pools**.

6.4. Specific pool properties of risk-sharing rules

The 'stand-alone for comonotic pools' property

- **Definition**: A RS rule $\mathbf{C} : \chi^n \rightarrow \chi^n$ satisfies **stand-alone property for comonotonic pools** if for any comonotonic pool $\mathbf{X} \in \chi^n$, one has that

$$\mathbf{C}(\mathbf{X}) = \mathbf{X}$$

- **Interpretation**: In a comonotonic pool, no diversification benefit arises from risk-sharing. Therefore, it may be reasonable to require that in such a pool each participant remains with his own risk.

6.4. Specific pool properties of risk-sharing rules

The 'stand-alone for comonotonic pools' property

- **Uniform RS**: **does not satisfy stand-alone for comonotonic pools**.
  Indeed, for the comonotonic pool $\mathbf{X} = (U, 2U, U)$, one has

$$C_1^{\text{uni}}(U, 2U, U) = \frac{4}{3}U \neq U$$

- **Conditional mean RS**: **satisfies stand-alone for comonotonic pools**.
  Indeed, for the comonotonic pool $\mathbf{X} = \left(g_1(S_{\mathbf{X}}), \ldots, g_n(S_{\mathbf{X}})\right)$, with all g_i non-decreasing functions, one has

$$C_i^{\text{cm}}(\mathbf{X}) = \mathbb{E}\left[X_i \mid S_{\mathbf{X}}\right] = \mathbb{E}\left[g_i(S_{\mathbf{X}}) \mid S_{\mathbf{X}}\right] = g_i(S_{\mathbf{X}}) = X_i$$

- **Definition**:

  $\mathbf{C}: \chi^n \rightarrow \chi^n$ satisfies **uniformity for exchangeable pools** if for any exchangeable pool $\mathbf{X} \in \chi^n$, one has:

$$\mathbf{C}(\mathbf{X}) = \frac{S_{\mathbf{X}}}{n}$$

- **Interpretation**:
  - The exchangeable pool $\mathbf{X}$ is **homogeneous** in the sense that the joint cdf of $\mathbf{X}$ is not changed by reshuffling.
  - In this case, the uniform RS rule seems to be appropriate.

- **Uniform RS**: **satisfies uniformity for exchangeable pools**.

  Indeed, for the exchangeable pool $\mathbf{X}$ one has that

$$C_i^{\text{uni}}(\mathbf{X}) = \frac{S_{\mathbf{X}}}{n}, \qquad i = 1, 2, \ldots, n$$

- **Proportional RS**: **satisfies uniformity for exchangeable pools**.

  Indeed, for the exchangeable pool $\mathbf{X}$, one has that

$$C_i^{\text{prop}}(\mathbf{X}) = \frac{\mathbb{E}\left[X_i\right]}{\mathbb{E}\left[S_{\mathbf{X}}\right]} S_{\mathbf{X}} = \frac{S_{\mathbf{X}}}{n}, \qquad i = 1, 2, \ldots, n$$

- **Order statistics RS**: **does not satisfy uniformity for exchangeable pools**.
  Indeed, for any exchangeable pool $\mathbf{X}$, one has that
  $$C_1^{\text{ord}}(\mathbf{X}) = \min\{X_1, \ldots, , X_n\}$$
  which is not necessarily equal to $\frac{S_{\mathbf{X}}}{n}$.
- **Conditional mean RS**: **satisfies uniformity for exchangeable pools**.
  Indeed, for any exchangeable pool $\mathbf{X}$ and any i one has that
  $$C_i^{\text{cm}}(\mathbf{X}) = \mathbb{E}[X_i \mid S_{\mathbf{X}}] = \frac{S_{\mathbf{X}}}{n}$$

7. Axiomatic characterization of the CMRS rule

- Consider the **CMRS rule**[22]:

$$\mathbf{C}^{\mathrm{cm}}(\mathbf{X}) = \mathbb{E}\left[\mathbf{X} \mid S_{\mathbf{X}}\right], \qquad i = 1, 2, \ldots, n$$

- We have proven that $\mathbf{C}^{\mathrm{cm}}$ satisfies the following properties:
  - $\mathbf{C}^{\mathrm{cm}}$ is an **aggregate RS rule**:

$$\mathbf{C}^{\mathrm{cm}}(\mathbf{X}) \text{ is } S_{\mathbf{X}} - \text{measurable}$$

  - $\mathbf{C}^{\mathrm{cm}}$ satisfies **no rip-off**: For any i,

$$C_i^{\mathrm{cm}}(\mathbf{X}) \leq F_{X_i}^{-1}(1)$$

  - $\mathbf{C}^{\mathrm{cm}}$ satisfies **actuarial fairness**: For any i,

$$\mathbb{E}\left[C_i^{\mathrm{cm}}(\mathbf{X})\right] = \mathbb{E}\left[X_i\right]$$

  - $\mathbf{C}^{\mathrm{cm}}$ satisfies **coalition fairness**: For any $\mathbf{Y} \in \mathcal{A}_{\mathbf{X}}$,

$$Y_i = X_i \Rightarrow C_i^{\mathrm{cm}}(\mathbf{Y}) = C_i^{\mathrm{cm}}(\mathbf{X})$$

[22] We consider the CMRS rule with $\chi = L_+^1$. However, the results in this section remain to hold true for any $\chi = L^q$ and $\chi = L_+^q$, with $q \in [1, \infty]$.

7. Axiomatic characterization of the CMRS rule

- **Theorem**:
  - Consider the RS rule $\mathbf{C} : \left(L^1_+\right)^n \to \left(L^1_+\right)^n$, with $n \geq 3$.
  - **C** is the **CMRS rule** if and only if it satisfies the following axioms:
    1. **C** is an **aggregate RS rule**.
    2. **C** satisfies **no rip-off**.
    3. **C** satisfies **actuarial fairness**.
    4. **C** satisfies **coalition fairness**.
- **Proof**:
  - $\Rightarrow$: See Denuit, Dhaene & Robert (2022).
  - $\Leftarrow$ [23] : See Jiao, Kou, Liu & Wang (2022). $\triangledown$

[23] This is the hard part of the proof.

7. Axiomatic characterization of the CMRS rule

- **Proposition**[24]:

  The axioms (1), (2), (3) and (4) are **independent**.

- Independence means that for any $n \geq 3$ and any choice of 3 of these axioms, there always exists a RS rule **C** which satisfies these 3 axioms, but not the 4th axiom.

[24] See J,K,L,W (2022).

7. Axiomatic characterization of the CMRS rule

- **Theorem**[25]:
  - Consider the RS rule $\mathbf{C} : \left(L^1_+\right)^n \rightarrow \left(L^1_+\right)^n$, with $n \geq 3$.
  - **C** is the **CMRS rule** if and only if it satisfies the following axioms:

    (1') **C** is an **aggregate RS rule**.

    (2') **C** satisfies **willingness-to-join**.

    (3') **C** satisfies **coalition fairness**.

- **Proof**:
  - $\Rightarrow$: Follows immediately.
  - $\Leftarrow$: Follows from property that 'willingness-to-join' implies 'actuarial fairness' and 'no rip-off'. $\bigtriangledown$

[25] See J,K,L,W (2022).

8. Further topics on DRS (not covered in this talk)

- **Quantile risk-sharing**[26]:
  - The QRS rule is defined such that the contribution of each participant i is given by $F_{X_i}^{-1}(p)$.
  - Probability level p is determined such that the full allocation condition is satisfied.
- **Pareto-optimal risk-sharing**[27]:
  - What is the relation between Pareto-optimal risk-sharing and comonotonicity?
- **Risk-sharing pool dynamics**[28]:
  - What if we add or withdraw one participant from the pool?[29]
  - What if the number of participants in the pool approaches infinity?

[26] See D,D,R (2022) and D,R C D (2023).
[27] see D,D,G,R (2023).
[28] Work in progress.
[29] See D,D,R (2022).

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